

DELIGNE PAIRING AND QUILLEN METRIC

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ABSTRACT. Let $X \rightarrow S$ be a smooth projective surjective morphism of relative dimension n , where X and S are integral schemes over \mathbb{C} . Let $L \rightarrow X$ be a relatively very ample line bundle. For every sufficiently large positive integer m , there is a canonical isomorphism of the Deligne pairing $\langle L, \dots, L \rangle \rightarrow S$ with the determinant line bundle $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ [PRS]. If we fix a hermitian structure on L and a relative Kähler form on X , then each of the line bundles $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ and $\langle L, \dots, L \rangle$ carries a distinguished hermitian structure. We prove that the above mentioned isomorphism between $\langle L, \dots, L \rangle \rightarrow S$ and $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ is compatible with these hermitian structures. This holds also for the isomorphism in [BSW] between a Deligne pairing and a certain determinant line bundle.

1. INTRODUCTION

Let S and X be integral schemes over \mathbb{C} , and let

$$f : X \rightarrow S$$

be a flat projective surjective morphism. Let n be the dimension of the fibers of f . Take algebraic line bundles $L_0, L_1, \dots, L_{n-1}, L_n$ over X . Their *Deligne pairing* is an algebraic line bundle

$$\langle L_0, \dots, L_n \rangle \rightarrow S$$

[Zh], [De]. If we equip the line bundles L_0, L_1, \dots, L_n with hermitian structure, then $\langle L_0, \dots, L_n \rangle$ gets a hermitian structure [Zh, Section 1.2], [De].

For any locally free sheaf F on X , we have the determinant line bundle

$$\text{Det}(F) = \det R^\bullet f_* F$$

over S [KM]. Given a virtual vector bundle $E - F$ on X , where E and F are vector bundles on X , there is the determinant line bundle $\text{Det}(E - F) := \text{Det}(E) \otimes \text{Det}(F)^*$ on S . This determinant Det is a homomorphism from the K -group of vector bundles on X to the Picard group of S .

Let L be a relatively very ample line bundle on X . Then for all sufficiently large positive integers m , there is a natural isomorphism of line bundles

$$(1.1) \quad \varphi : \text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m}) \rightarrow \langle L, \dots, L \rangle$$

[PRS, Theorem 3].

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Assuming that the above morphism f is smooth, if we fix hermitian structures on the above two vector bundles E and F , and also fix a relative Kähler form on X , then the line bundle $\text{Det}(E - F)$ gets a hermitian structure [BGS], [Qu].

Our aim here is to prove the following (see Theorem 5.2):

Theorem 1.1. *Let $f : X \longrightarrow S$ be a smooth projective surjective morphism between integral schemes over \mathbb{C} . Let (L, h) be a relatively very ample hermitian line bundle on X . Fix a relative Kähler form on X , and also equip \mathcal{O}_X with the hermitian structure for which the constant function 1 has point-wise norm 1. Let h^Q and h^D be the hermitian structures on the line bundles $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ and $\langle L, \dots, L \rangle$ respectively. Then there is a positive real number c such that*

$$\varphi^* h^D = c \cdot h^Q,$$

where φ is the isomorphism in (1.1).

The constant c is compatible with base change.

The constant c is independent of the hermitian structure h .

It should be emphasized that the constant c in Theorem 1.1 may depend on the family f and also on the Néron–Severi class of the restriction of L to a fiber of f . However we do not know explicit examples. It would be interesting to be able to construct explicit examples with different values of c . It should be mentioned that the proof of Theorem 1.1 relies on curvature computations. Therefore, the constant c in Theorem 1.1 cannot be computed using the method of proof given here.

The construction of h^D requires only a hermitian structure on L . In particular, it does not require to have a relative Kähler form on X . Although in general the Quillen metric on a determinant line bundle depends on the choice of a relative Kähler form, it turns out that the hermitian structure h^Q on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ is independent of the choice of the relative Kähler form on X ; see Proposition 3.1.

The proof of Theorem 1.1 is based on the following:

- There are no nonconstant harmonic functions on a compact connected complex manifold.
- The metric on a Deligne pairing of line bundles over a smooth variety has some regularity as the variety degenerates [Mo].
- The Quillen metric for hermitian vector bundles over a smooth variety can be controlled as the variety degenerates [Bi], [Yo].

2. COMPARISON OF CURVATURES

2.1. Deligne pairing. Let $f : X \longrightarrow S$ be a smooth projective surjective morphism between integral schemes over \mathbb{C} . The relative dimension of this projection will be denoted by n . Fix $n + 1$ line bundles L_0, \dots, L_n on X . The Deligne pairing $\langle L_0, L_1, \dots, L_n \rangle$ is a line bundle on the base S that defines a map

$$(\mathcal{P}ic(X))^{n+1} \longrightarrow \mathcal{P}ic(S),$$

where $\mathcal{P}ic$ denotes the stack of line bundles, which is multilinear and invariant under the permutations of the factors, [De], [Zh]. We recall that the above the line bundle $\langle L_0, L_1, \dots, L_n \rangle$ is locally generated by symbols of the form $\langle \sigma_0, \sigma_1, \dots, \sigma_n \rangle$, where σ_i , $0 \leq i \leq n$, is a rational section of the line bundle L_i whose divisor $\text{Div}(\sigma_i)$ has the property that

$$\bigcap_{i=0}^n \text{Div}(\sigma_i) = \emptyset.$$

The transition functions for the above local trivializations satisfy the following condition: if

$$\mathcal{D} := \bigcap_{i \neq k} \text{Div}(\sigma_i)$$

is flat over S , then

$$\langle \sigma_0, \dots, \zeta \sigma_k, \dots, \sigma_n \rangle = \zeta[\mathcal{D}] \cdot \langle \sigma_0, \dots, \sigma_n \rangle$$

for any rational function ζ , where $\zeta[\mathcal{D}] = \text{Norm}_{\mathcal{D}/S}(\zeta)$ is the norm function. Using the two properties of Deligne pairing mentioned earlier, it can be shown that this condition uniquely determines the transition functions. The key input for this is the Weil reciprocity formula.

For each $0 \leq i \leq n$, fix a C^∞ hermitian structure h_i on the line bundle L_i . Given this data, there is a hermitian structure $\langle h_0, \dots, h_n \rangle$ on the Deligne pairing $\langle L_0, \dots, L_n \rangle$. This hermitian structure $\langle h_0, \dots, h_n \rangle$ satisfies the following condition: Take the hermitian structure $h'_0 := h_0 \exp(-\mu)$ on L_0 , where μ is a C^∞ real valued function on X . Then

$$\langle h'_0, \dots, h_n \rangle = \langle h_0, \dots, h_n \rangle \cdot \exp(-\widehat{\mu}),$$

where $\widehat{\mu}$ is the real valued smooth function

$$\left(\frac{\sqrt{-1}}{2\pi} \right)^n \int_{X/S} \mu \cdot (\partial \bar{\partial} \log h_1) \wedge \dots \wedge (\partial \bar{\partial} \log h_n).$$

2.2. The curvature. Fix a relative Kähler form on X . Fix a holomorphic hermitian line bundle (L, h) on X . Equip \mathcal{O}_X with the standard hermitian structure for which the constant function 1 has point-wise norm 1. Take any integer m . Let h^Q be the hermitian structure on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$. The hermitian structure on $\langle L, \dots, L \rangle$ will be denoted by h^D .

Proposition 2.1. *The curvature of the hermitian metric h^D on $\langle L, \dots, L \rangle$ coincides with the curvature of the Quillen metric h^Q on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$.*

Proof. The Chern form of the metric h^D on $\langle L, \dots, L \rangle$ coincides with the fiber integral

$$(2.1) \quad \int_{X/S} c_1(L, h)^{n+1}$$

(see [Zh, p. 81, Section 1.2]). On the other hand, a theorem of Bismut, Gillet and Soulé, [BGS, Theorem 0.1], says that the Chern form of the determinant line bundle

$$(\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m}), h^Q) \longrightarrow S$$

is the degree two component of the Riemann-Roch fiber integral
(2.2)

$$c_1(\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m}, h^Q) = \left(\int_{X/S} ch(L - \mathcal{O}_X, h)^{n+1} \cdot ch(L, h)^m \cdot Td(X/S) \right)_{(2)},$$

where Td is the Todd form. This result of [BGS] was extended to (smooth) Kähler fibrations over singular base spaces in [FS, § 12]. Since the base S is reduced and irreducible, we may choose a desingularization S' of S and consider the pulled back family $X \times_S S'$ over S' . Let (L', h') be the pullback of (L, h) to $X \times_S S'$. Then (2.2) for L' coincides with the pullback of (2.2) to S' .

Note that

$$ch(L - \mathcal{O}_X, h) = c_1(L, h) + \text{higher order terms}.$$

Hence the only contribution of the differential form $ch(L, h)^m \cdot Td(X/S)$ in the integral in (2.2) is the constant 1, and also the higher order terms in $ch(L - \mathcal{O}_X, h)$ do not contribute in the integral. Consequently, the integral in (2.2) coincides with the one in (2.1). \square

3. INDEPENDENCE FROM THE RELATIVE KÄHLER STRUCTURE

Let $f : X \rightarrow S$ be a smooth projective surjective morphism of relative dimension n , where X and S are integral schemes over \mathbb{C} . Fix a relative Kähler form $\omega_{X/S}$ on X . Let L be a line bundle on X equipped with a hermitian structure. As before, fix the hermitian metric on \mathcal{O}_X for which the constant function 1 has point-wise norm 1. Let h^Q be Quillen metric on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$.

Proposition 3.1. *The Quillen metric h^Q on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ does not depend on the relative Kähler form $\omega_{X/S}$.*

Proof. In view of the base change property of h^Q , it suffices to prove the proposition for the absolute case, meaning when S is a single point.

So, let (X, ω_X) be a compact connected Kähler manifold, and let (L, h) be a holomorphic hermitian line bundle on it. Let ξ denote the virtual vector bundle $(L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m}$ equipped with the (virtual) hermitian structure h_ξ given by h and the hermitian metric on \mathcal{O}_X for which the constant function 1 has point-wise norm 1. Then the Chern character form for h_ξ is

$$(3.1) \quad ch(\xi, h_\xi) = c_1(L, h)^{n+1} + \text{higher order terms}.$$

Let ω'_X be another Kähler form on X . Bott and Chern, [BC], defined a class of differential forms $\widetilde{Td}(\omega_X, \omega'_X)$, unique in the space of forms modulo the images of ∂_X and $\bar{\partial}_X$, such that

$$\frac{1}{2\pi\sqrt{-1}} \partial_X \bar{\partial}_X \widetilde{Td}(\omega_X, \omega'_X) = Td(X, \omega_X) - Td(X, \omega'_X).$$

Let h^Q (respectively, h'^Q) be the Quillen metric on the determinant line bundle (now it is just a complex line as S is only a point) $\text{Det}((L - \mathcal{O}_X)^{\otimes n} \otimes L^{\otimes m})$ for the Kähler metric

ω_X (respectively, ω'_X). A theorem of Bismut, Gillet and Soulé, [BGS, Theorem 0.2], says that

$$(3.2) \quad \log \left(\frac{h^Q}{h'^Q} \right) = \int_X \widetilde{Td}(\omega_X, \omega'_X) ch(\xi, h_\xi).$$

Only the term in degree $2n$ of the above integrand contributes, and by (3.1), this term vanishes. Therefore,

$$\log \left(\frac{h^Q}{h'^Q} \right) = 0.$$

This implies that $h^Q = h'^Q$. This completes the proof of the lemma. \square

We fix a complex projective manifold X . Let $\text{NS}(X)$ be the Néron–Severi group $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ of X . For an element $\nu \in \text{NS}(X)$ of X , let $P = \text{Pic}^\nu(X)$ be the connected component of the Picard group of X corresponding to ν . We choose ν such that all line bundles on X lying in P are very ample. Take any $L \in P$, and take any hermitian structure h_L on L . Let h_L^Q and h_L^D be the hermitian structures on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ and $\langle L, \dots, L \rangle$ respectively.

Proposition 3.2. *There exists a real number $c_\nu > 0$, that depends only on X and $\nu \in \text{NS}(X)$, such that for any L and h_L as above, the following equality of hermitian structures on $\text{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ holds:*

$$(3.3) \quad c_\nu \cdot h_L^Q = \varphi^* h_L^D,$$

where φ is the isomorphism in (1.1). In particular, c_ν is independent of the point $L \in \text{Pic}^\nu(X)$ and the hermitian structure h_L .

Proof. Fix a Poincaré line bundle $\mathcal{L} \rightarrow X \times P$. Let h be a hermitian structure on \mathcal{L} . Consider the trivial family $X \times P \rightarrow P$. Let

$$\text{Det}((\mathcal{L} - \mathcal{O}_{X \times P})^{\otimes(n+1)} \otimes \mathcal{L}^{\otimes m}) \rightarrow P$$

be the determinant line bundle. Let h^Q and h^D be the hermitian structures on the line bundles $\text{Det}((\mathcal{L} - \mathcal{O}_{X \times P})^{\otimes(n+1)} \otimes \mathcal{L}^{\otimes m})$ and $\langle \mathcal{L}, \dots, \mathcal{L} \rangle$ respectively. The curvatures of h^Q and h^D coincide by Proposition 2.1. Consequently,

$$\log \left(\frac{\varphi^* h^D}{h^Q} \right)$$

is a harmonic function on P , where φ is the isomorphism in (1.1). On the other hand, there are no nonconstant harmonic functions on P because it is compact and connected. Therefore, $(\varphi^* h^D)/h^Q$ is a constant function on P .

For any point $L \in P$, any hermitian structure on $\mathcal{L}|_{X \times \{L\}} = L$ can be extended to a hermitian structure on \mathcal{L} . Therefore, to prove the proposition, it suffices to show that $(\varphi^* h_L^D)/h_L^Q$ is independent of the hermitian structure on L .

To prove that $(\varphi^* h_L^D)/h_L^Q$ is independent of the hermitian structure on L , take any two hermitian structures h_L^1 and h_L^2 on L . Let Z be a compact connected complex manifold of positive dimension. Fix two distinct points z_0 and z_1 of Z . Consider the line bundle

$$p_X^* L \rightarrow X \times Z,$$

where p_X is the natural projection of $X \times Z$ to X . Let h_1 and h_2 be two hermitian structures on p_X^*L such that

- (1) the restriction of h_1 (respectively h_2) to $(p_X^*L)|_{X \times \{z_0\}}$ is h_L^1 (respectively h_L^2), and
- (2) the restrictions of h_1 and h_2 to $(p_X^*L)|_{X \times \{z_1\}}$ coincide.

Let h_1^Q (respectively, h_2^Q) be the hermitian metric on

$$\text{Det}((p_X^*L - \mathcal{O}_{X \times Z})^{\otimes(n+1)} \otimes (p_X^*L)^{\otimes m}) \longrightarrow Z$$

for h_1 (respectively h_2). Similarly, let h_1^D (respectively, h_2^D) be the hermitian metric on

$$\langle p_X^*L, \dots, p_X^*L \rangle \longrightarrow Z$$

for h_1 (respectively h_2). As before, there is a constant d_1 (respectively, d_2) such that

$$d_1 \cdot h_1^Q = \varphi^* h_1^D \quad (\text{respectively, } d_2 \cdot h_2^Q = \varphi^* h_2^D),$$

because there are no nonconstant harmonic functions on the compact connected complex manifold Z (the isomorphism φ is as in (1.1)). Note that we are using that the curvature of h_1^Q (respectively, h_2^Q) coincides with that of h_1^D (respectively, h_2^D) by Proposition 2.1. Since the restriction of h_1 to $(p_X^*L)|_{X \times \{z_1\}}$ coincides, by construction, with the restriction of h_2 to $(p_X^*L)|_{X \times \{z_1\}}$, we now conclude that $d_1 = d_2$ using base change. \square

Henceforth, we will identify the two sides of (1.1) using the isomorphism φ . In particular, we will omit referring explicitly to φ .

4. COMPARISON OF METRICS: THE CASE OF RIEMANN SURFACES

The above argument that uses Proposition 2.1 combined with the fact that there are no nonconstant harmonic functions on a compact complex manifold, can be extended to line bundles on families of Riemann surfaces.

Proposition 4.1. *Let X be a compact connected Riemann surface of genus p with $p \geq 3$, and let L be a holomorphic hermitian line bundle on X of degree d , with $d \geq 2p - 1$. Let h^Q and h^D be the hermitian structures on $\text{Det}((L - \mathcal{O}_X)^{\otimes 2} \otimes L^{\otimes m})$ and $\langle L, L \rangle$ respectively. Then the positive real number c that satisfies the identity*

$$c \cdot h^Q = h^D$$

depends only on p and d . In other words, c is independent of both the conformal structure of X and the holomorphic structure of L .

Proof. The above assumption that $d \geq 2p - 1$ is needed to ensure that the line bundles are very ample. This enables us to invoke [PRS, Theorem 3].

From Proposition 3.2 and Proposition 3.1 we know the constant c in the proposition depends only on the degree d and the complex structure of the Riemann surface, and it is independent of the metric on X .

Let \mathcal{M}_p be the moduli space of compact Riemann surfaces of genus p . Let $\widetilde{\mathcal{M}}_p$ be the Satake compactification of \mathcal{M}_p . We recall that $\widetilde{\mathcal{M}}_p$ is defined to be the closure of \mathcal{M}_p in the Satake compactification of the moduli space of principally polarized Abelian

varieties [Ba]. The Satake compactification $\widetilde{\mathcal{M}}_p$ is an irreducible projective variety, and the complement

$$\widetilde{\mathcal{M}}_p \setminus \mathcal{M}_p \subset \widetilde{\mathcal{M}}_p$$

is of codimension at least two [Ba], [HM, p. 45]. This codimension is one if $p = 2$. The assumption in the proposition that $p \geq 3$ is needed here.

Consequently, there is a dense subset $U \subset \mathcal{M}_p$ in Euclidean topology satisfying the following condition: for any two points $x, y \in U$, there are compact Riemann surfaces C_1, \dots, C_ℓ , and holomorphic maps

$$\phi_i : C_i \longrightarrow \mathcal{M}_p,$$

such that

- $\phi_i(C_i) \subset U$ for all $1 \leq i \leq \ell$,
- $\bigcup_{i=1}^\ell \phi_i(C_i)$ is connected, and
- $\{x, y\} \subset \bigcup_{i=1}^\ell \phi_i(C_i)$.

The constant c is clearly continuous over \mathcal{M}_p . Therefore, to prove the proposition, it is enough to show that for any holomorphic map

$$\phi : C \longrightarrow \mathcal{M}_p,$$

where C is some compact connected Riemann surface, the constant c in the proposition remains unchanged as the Riemann surface X moves over $\phi(C) \subset \mathcal{M}_p$ (with d fixed).

Take (C, ϕ) as above. There is a finite (possibly ramified) covering

$$\gamma : \widetilde{C} \longrightarrow C$$

such that there is an algebraic family of Riemann surfaces over \widetilde{C} represented by the morphism

$$\phi \circ \gamma : \widetilde{C} \longrightarrow \mathcal{M}_p.$$

In other words, $\phi \circ \gamma$ is the classifying map for this family. It may be mentioned that we may take \widetilde{C} to be the parameter space for Riemann surfaces corresponding to points in $\phi(C)$ equipped with a level structure.

In view of Proposition 2.1, we conclude that the quotient $\log(h^Q/h^D)$ is constant for the above family over \widetilde{C} , because there are no nonconstant harmonic functions on \widetilde{C} . Therefore, h^Q/h^D is a constant. As explained above, this completes the proof of the proposition. \square

5. COMPARISON OF METRICS: THE GENERAL CASE

The following is the key lemma.

Lemma 5.1. *Let \overline{C} be a smooth complete connected complex curve, and let $C \subset \overline{C}$ be the complement finitely many points. Let $f : \overline{\mathcal{X}} \longrightarrow \overline{C}$ be a flat projective family of n -dimensional varieties, which is smooth with connected fibers over C . Let $\overline{\mathcal{L}}$ be a relatively*

very ample line bundle over $\overline{\mathcal{X}}$ equipped with a hermitian structure. Define $\mathcal{X} := f^{-1}(C)$, and $\mathcal{L} := \overline{\mathcal{L}}|_{\mathcal{X}}$. Then there is a positive real number c such that

$$c \cdot h^Q = h^D,$$

where h^Q is the Quillen metric on the line bundle $\text{Det}((\mathcal{L} - \mathcal{O}_{\mathcal{X}})^{\otimes(n+1)} \otimes (\mathcal{L})^{\otimes m})$ and h^D is the metric on the Deligne pairing $\langle \mathcal{L}, \dots, \mathcal{L} \rangle$ (recall that the line bundles are identified by (1.1)).

Proof. From Proposition 2.1 we know that $\log(h^Q/h^D)$ is harmonic on C . A theorem of Moriwaki says that the hermitian structure h^D possesses a continuous extension to the line bundle

$$\langle \overline{\mathcal{L}}, \dots, \overline{\mathcal{L}} \rangle \longrightarrow \overline{C}$$

(see [Mo, Theorem A]).

The degeneration of the Quillen metric for semistable families was first computed by Bismut in [Bi, (0.7)], and this was generalized by Yoshikawa in [Yo, Theorem 1.1]. In order to apply this result, first the semistable reduction theorem of Mumford, [KKMS, p. 54], is used. It yields the existence of a finite morphism

$$\nu : \tilde{C} \longrightarrow \overline{C}$$

with the following property: The desingularization

$$Z \longrightarrow \overline{\mathcal{X}} \times_{\overline{C}} \tilde{C}$$

gives rise to a family $Z \longrightarrow \tilde{C}$, which is smooth over $\nu^{-1}(C)$ and whose fibers over points of $\nu^{-1}(\overline{C} \setminus C)$ are simple normal crossings divisors. The projective variety Z is now equipped with the restriction of a Fubini–Study metric (with respect to some projective embedding).

Let

$$\phi : Z \longrightarrow \tilde{C}$$

be the natural projection. Let

$$(5.1) \quad \eta : Z \longrightarrow \overline{\mathcal{X}}$$

be the obvious morphism.

According to [Bi, Theorem 0.1, (0.6)], the Chern form of the Quillen metric over C extends as the sum of two currents on \tilde{C} . The first current is in $L_{loc}^r(\tilde{C})$ with $1 \leq r < 2$, and the second one is equal to

$$(5.2) \quad -\frac{1}{2} \left[\int_{\Sigma} Td(T(\Sigma/\tilde{C})) E(\alpha) ch(\xi) \right]^{(0)} \delta_{\Delta},$$

where

- $\Delta := (\nu \circ \phi)^{-1}(\overline{C} \setminus C)$ is the singular fiber,
- $\Sigma \subset \Delta$ is the singular locus of Δ ,
- α is the normal bundle of Σ in Δ ,

- ξ is the virtual vector bundle $(\eta^*\overline{\mathcal{L}} - \mathcal{O}_Z)^{\otimes(n+1)} \otimes (\eta^*\overline{\mathcal{L}})^{\otimes m}$ equipped with the (virtual) hermitian structure h_ξ , defined by the hermitian metric on $\overline{\mathcal{L}}$ and the standard hermitian metric on \mathcal{O}_Z , and
- E is the additive arithmetic genus that is generated by a certain holomorphic function on \mathbb{C} .

By (3.1), the integral in (5.2) vanishes. Take any boundary point $z_0 \in \nu^{-1}(\overline{C} \setminus C)$; fix a local holomorphic coordinate s on an open neighborhood $U_{z_0} \subset \tilde{C}$ of z_0 such that $s(z_0) = 0$. Since the function $\log(h^Q/h^D) \circ \nu$ on $U_{z_0} \setminus \{z_0\}$ is harmonic, it is of the form

$$a + \gamma \log |s| \cdot \operatorname{Re}(\psi),$$

where a and γ are constants, and ψ is a holomorphic function on $U_{z_0} \setminus \{z_0\}$. Since the Chern current of h^Q is in L^1_{loc} , we conclude that

- (1) $\gamma = 0$, and
- (2) ψ extends holomorphically across z_0 .

Note that h^D , considered as function after choosing a trivialization of $\langle \overline{\mathcal{L}}, \dots, \overline{\mathcal{L}} \rangle$ over U_{z_0} , is continuous and nowhere vanishing on U_{z_0} .

Consequently, $\log(h^Q/h^D) \circ \nu$ extends as a harmonic function from $\nu^{-1}(C)$ to \tilde{C} . Therefore, $\log(h^Q/h^D)$ is a constant function.

However, the above theorem of Bismut applies to singular fibers, where at most two components meet at any point. The methods of Bismut were extended to the general case of arbitrary singular fibers of flat projective morphisms by Yoshikawa in [Yo]. Yoshikawa's main theorem, [Yo, Theorem 1.1], implies that the Quillen metric extends continuously up to a certain summand, which consist of an integral containing the Chern character form $ch(\xi, h_\xi)$ over an n -dimensional variety. Again the integral vanishes by (3.1). This shows that both metrics h^Q and h^D on the determinant line bundle extend continuously to \tilde{C} ; here we are using that the line bundles

$$\operatorname{Det}((\eta^*\overline{\mathcal{L}} - \mathcal{O}_Z)^{\otimes(n+1)} \otimes (\eta^*\overline{\mathcal{L}})^{\otimes m}) \longrightarrow \tilde{C} \quad \text{and} \quad \langle \eta^*\overline{\mathcal{L}}, \dots, \eta^*\overline{\mathcal{L}} \rangle \longrightarrow \tilde{C}$$

are canonically isomorphic (see (1.1), [PRS, Theorem 3]). Consequently, the quotient $\log(h^Q/h^D) \circ \nu$ is a continuous harmonic function of \tilde{C} , implying that $\log(h^Q/h^D)$ is a constant. \square

The following theorem was mentioned in the introduction (Theorem 1.1) as the main result.

Theorem 5.2. *Let $f : X \longrightarrow S$ be a smooth projective surjective morphism between integral schemes over \mathbb{C} . Let (L, h) be a relatively very ample hermitian line bundle on X . Fix a relative Kähler form on X , and also equip \mathcal{O}_X with the natural hermitian structure. Let h^Q and h^D be the hermitian structures on the line bundles $\operatorname{Det}((L - \mathcal{O}_X)^{\otimes(n+1)} \otimes L^{\otimes m})$ and $\langle L, \dots, L \rangle$ respectively. Then there is a positive real number c such that*

$$\varphi^* h^D = c \cdot h^Q,$$

where φ is the isomorphism in (1.1).

The constant c is compatible with base change.

The constant c is independent of the hermitian structure h .

Proof. In view of the base change property of the isomorphism in (1.1), the first part of the theorem follows from Lemma 5.1. To explain this, take any morphism

$$\rho : C \longrightarrow S,$$

where C is a smooth complex curve such that

- the pullback $C \times_S X$ extends to a flat family of projective varieties over the smooth compactification \overline{C} of C , and
- the pullback of L to $C \times_S X$ extends to a relative very ample line bundle for the above family over \overline{C} .

From Lemma 5.1 and the base change property of the isomorphism in (1.1) we have the following. There is a positive real number c_ρ such that for any point $z \in C$, the equality

$$(\varphi^* h^D)_{\rho(z)} = c_\rho \cdot (h^Q)_{\rho(z)}$$

holds. Now we define an equivalence relation on the points on S . Two points z and z' of S will be called equivalent if there are finitely many morphisms ρ_i of the above type such that the union $\bigcup_i \text{image}(\rho_i)$ is connected and contains both z and z' . From the above observation and the base change property of the isomorphism in (1.1) we conclude that for any equivalence class \mathcal{C} , there is a positive real number $c_{\mathcal{C}}$ such that for every point $z \in \mathcal{C}$, we have

$$(\varphi^* h^D)_z = c_{\mathcal{C}} \cdot (h^Q)_z.$$

Since there exists an equivalence class which is dense in S in the Euclidean topology, the first part of the theorem follows.

The determinant line bundle, the Quillen metric, the Deligne pairing and the hermitian structure on the Deligne pairing are all compatible with base change. The isomorphism φ in (1.1) is also compatible with base change. Therefore, the second part of the theorem follows.

The third part follows from Proposition 3.2. □

6. FURTHER APPLICATION OF THE METHOD

We recall Theorem 5.8 in page 371 of [ACG]:

Theorem 6.1 ([ACG]). *Let $f : X \longrightarrow S$ be an algebraic family of nodal complex projective curves. Let L and M be two algebraic line bundles on X . There is a canonical isomorphism*

$$\langle L, M \rangle \xrightarrow{\sim} \text{Det}(L \otimes M) \otimes \text{Det}(L)^* \otimes \text{Det}(M)^* \otimes \text{Det}(\mathcal{O}_X) = \text{Det}((L - \mathcal{O}_X)(M - \mathcal{O}_X))$$

compatible with base change.

Theorem 6.1 generalizes to higher dimensions. To explain this, let $X \rightarrow S$ be a flat projective surjective morphism, of relative dimension n , between integral schemes over \mathbb{C} . Fix line bundles L_0, L_1, \dots, L_n on X .

Theorem 6.2 ([BSW]). *There is a canonical isomorphism of the Deligne pairing*

$$\langle L_0, \dots, L_n \rangle \rightarrow S$$

and the determinant line bundle $\text{Det}(\bigotimes_{i=0}^n (L_i - \mathcal{O}_X))$, which is compatible with base change.

For each $0 \leq i \leq n$, fix a hermitian structure h_i on L_i . This produces a hermitian structure h^D on the line bundle $\langle L_0, \dots, L_n \rangle \rightarrow S$. Now fix a relative Kähler form $\omega_{X/S}$ on X . This and the hermitian metrics h_i together define a hermitian structure h^Q on the line bundle $\text{Det}(\bigotimes_{i=0}^n (L_i - \mathcal{O}_X))$ [Qu], [BGS].

It can be shown that h^Q is independent of the choice of the relative Kähler form $\omega_{X/S}$. Indeed, its proof is exactly identical to that of Proposition 3.1. In fact the following holds:

Theorem 6.3. *There is a positive real number c such that the isomorphism in Theorem 6.2 takes the hermitian structure h^D to $c \cdot h^Q$.*

The proof of Theorem 6.3 is same as that of Theorem 5.2.

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